

# Two dimensional disjoint minimal graphs

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## Abstract

In this paper, under the assumption of Gauss curvature vanishing at infinity, we will prove Meeks' conjecture: the number of disjointly supported minimal graphs in  $\mathbb{R}^3$  is at most two.

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## 1 Introduction

Let  $\Omega$  be an open subset in  $\mathbb{R}^2$  and denote its boundary by  $\partial\Omega$ . As we know, if a function  $u(x)$  which is defined on  $\Omega$  satisfies the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \quad (1)$$

$G = \{(x, u(x)) | x \in \Omega\}$  is called a minimal graph in  $\mathbb{R}^3$ . Furthermore, we call the minimal graph  $G$  is supported on  $\Omega$  if  $u|_{\partial\Omega} = 0$  and  $u \geq 0$ .

Meeks [4] has conjectured that the number of disjointly supported minimal graphs with zero boundary values over an open subset in  $\mathbb{R}^2$  is at most 2. In fact, for arbitrary dimension, Meeks-Rosenberg [3] proved if a set of disjointly supported minimal graphs have bounded gradient, then the number of the graphs must be finite. Later, Li-Wang [2] gave an upper bound of the number of the graphs without any assumption on the growth rate of each graph. As a corollary, when minimal graphs are two dimensional in  $\mathbb{R}^3$ , they obtained the number is at most 24. At the same time, Spruck [6] proved that there are at most two admissible sub-linear growth solution pairs of the equation (1) defined over disjoint domains. Recently, by using angular density, Tkachev [7] showed the number of two dimensional disjointly supported minimal graphs is less than or equals 3.

Observing the similarity between the disjoint  $d$ -massive set and disjointly supported minimal graphs, we can apply the method of proving finiteness theorem of disjoint  $d$ -massive sets in  $\mathbb{R}^2$  [1] to study disjoint minimal graphs. Actually, we obtain the following theorem:

**Theorem 1.1.** *Suppose  $\{G_i = (\Omega_i, u_i)\}_{i=1}^k$  is a set of disjointly supported minimal graphs in  $\mathbb{R}^3$  where each  $\Omega_i$  is an open subset in  $\mathbb{R}^2$ . If the Gauss curvature  $K_i(x)$  of each graph satisfies*

$$K_i(x) \rightarrow 0 \quad (|x| \rightarrow \infty),$$

then the number of the graphs  $k$  is at most two.

By choosing a slight different region of integration, one has a stronger result comparing with the theorem of Spruck[6].

**Corollary 1.2.** *Suppose  $\{G_i = (\Omega_i, u_i)\}_{i=1}^k$  is a set of disjointly supported minimal graphs in  $\mathbb{R}^3$  where each  $\Omega_i$  is an open subset in  $\mathbb{R}^2$ . If each graph has sub-linear growth, then  $k$  is at most two.*

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## 2 Proof of theorem 1.1

In the following, we denote the 3-dimensional ball of radius  $R$  centered at the origin of  $\mathbb{R}^3$  by  $B^3(R)$  and the 2-dimensional sphere of radius  $R$  by  $S^2(R)$ . Actually, the key is to establish a refined estimate of the sum of all curves' length  $\ell(G_i \cap S^2(R))$  when  $R$  is sufficiently large.

**Theorem 2.1.** *Suppose  $\{G_i = (\Omega_i, u_i)\}_{i=1}^k$  is a set of disjointly supported minimal graphs in  $\mathbb{R}^3$  where the Gauss curvature  $K_i(x)$  of each  $G_i$  satisfies*

$$K_i(x) \rightarrow 0 \quad (|x| \rightarrow \infty).$$

*For a sufficiently large radius  $R$ ,  $\sum_{i=1}^k \ell(G_i \cap S^2(R))$  is bounded by*

$$\sum_{i=1}^k \ell(G_i \cap S^2(R)) \leq \pi^2 R + o(1)R.$$

*In particular, when  $k = 3$ , we have an refined estimate*

$$\sum_{i=1}^3 \ell(G_i \cap S^2(R)) \leq 2\sqrt{2}\pi R + o(1)R.$$

Before proving the theorem 2.1, we introduce a lemma.

**Lemma 2.2.** *Let  $B_+^3(R)$  be a 3-dimensional upper half ball with the radius  $R$  and  $S_+^2(R)$  be a 2-dimensional upper half sphere. Suppose  $\pi_i : G_i \rightarrow \mathbb{R}^2$  is the natural projective map. If  $\Sigma_1, \Sigma_2, \dots, \Sigma_s$  are the planes in  $\mathbb{R}^3$  such that each interior of  $\pi_i(\Sigma_i \cap B_+^3(R))$  does not intersect for a sufficiently large  $R$ , then the length of the curve  $\Sigma_i \cap S_+^2(R)$  satisfies*

$$\sum_{i=1}^s \ell(\Sigma_i \cap S_+^2(R)) \leq \pi^2 R.$$

Moreover, when  $s = 3$ , we have a refined estimate

$$\sum_{i=1}^3 \ell(\Sigma_i \cap S_+^2(R)) \leq 2\sqrt{2}\pi R.$$

*Proof.* Suppose  $D(R) = \{(x_1, x_2, 0) | x_1^2 + x_2^2 \leq R^2\}$  is a disk in  $\mathbb{R}^3$ . Since each  $\Sigma_i$  is a plane,  $\Sigma_i \cap D(R)$  is a chord and let  $\theta_i$  be the corresponding central angle. Here we only need to consider the case that the union of each chords  $\cup_{i=1}^s (\Sigma_i \cap D(R))$  is a polygon. Otherwise, one can add more planes which still satisfy the required conditions such that above intersection yields a polygon.

If the centre of the disk  $D(R)$  is in the interior of the polygon or on one of the edge of the polygon, this means each central angle  $\theta_i$  satisfies  $0 < \theta_i \leq \pi$ . For each interior of  $\pi_i(\Sigma_i \cap B_+^3(R))$  does not intersect, by a simple computation one can obtain the following inequality about the length of the arc  $\ell(\Sigma_i \cap S_+^2(R))$

$$\ell(\Sigma_i \cap S_+^2(R)) \leq \pi R \sin \frac{\theta_i}{2}.$$

The RHS achieves the maximum if and only if  $\Sigma_i$  is perpendicular to the disk  $D(R)$ . Thus

$$\begin{aligned} \sum_{i=1}^s \ell(\Sigma_i \cap S_+^2(R)) &\leq \sum_{i=1}^s \pi R \sin \frac{\theta_i}{2} \leq \pi R s \sin\left(\frac{1}{s} \sum_{i=1}^s \frac{\theta_i}{2}\right) \\ &\leq \pi R s \sin\left(\frac{\pi}{s}\right) \leq \pi^2 R. \end{aligned} \quad (2)$$

In the second above inequality, we use the concave property of the sine function on the interval  $[0, \pi]$ .

For a special case when  $s = 3$ , from (2) one can yield

$$\sum_{i=1}^3 \ell(\Sigma_i \cap S_+^2(R)) \leq 3\pi R \sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}\pi R. \quad (3)$$

If the centre of the disk  $D(R)$  is outside the polygon, namely there exists an  $i_0$  such that  $\theta_{i_0} > \pi$ . For simplicity, let us assume  $i_0 = s$ . A similar computation induces that

$$\begin{aligned} \ell(\Sigma_i \cap S_+^2(R)) &\leq \pi R \sin \frac{\theta_i}{2} \quad \text{for } 1 \leq i \leq s-1, \\ \ell(\Sigma_s \cap S_+^2(R)) &\leq R\theta_s. \end{aligned}$$

The first equality holds if and only if  $\Sigma_i$  is perpendicular to the disk and the second equality holds if and only if  $\Sigma_s$  is in the same plane of the disk  $D(R)$ . Hence one will have

$$\begin{aligned} \sum_{i=1}^s \ell(\Sigma_i \cap S_+^2(R)) &\leq \sum_{i=1}^{s-1} \pi R \sin \frac{\theta_i}{2} + R\theta_s \leq \sum_{i=1}^{s-1} \pi R \sin \frac{\theta_i}{2} + 2\pi R \sin \frac{\theta_s}{4} \\ &\leq \pi R(s+1) \sin \frac{\pi}{s+1} \leq \pi^2 R. \end{aligned} \quad (4)$$

If  $s = 3$ , by (4) we obtain that

$$\sum_{i=1}^3 \ell(\Sigma_i \cap S_+^2(R)) \leq 4\pi R \sin\left(\frac{\pi}{4}\right) = 2\sqrt{2}\pi R \quad (5)$$

The conclusion is derived from (2), (4) and (3), (5).  $\square$

*Proof of the theorem 2.1.* For each minimal graph  $G_i$ , since the Gauss curvature  $K_i = 0$  at infinity, it means  $G_i$  is asymptotic to a flat plane. Therefore, we can use the intersection of a plane  $\Sigma_i$  and  $S_+^2(R)$  to approximate the curve  $G_i \cap S^2(R)$ . By the lemma 2.2, one has

$$\ell(G_i \cap S^2(R)) \leq \ell(\Sigma_i \cap S_+^2(R)) + o(1)R.$$

Therefore

$$\sum_{i=1}^k \ell(G_i \cap S^2(R)) \leq \sum_{i=1}^k \ell(\Sigma_i \cap S_+^2(R)) + o(1)R \leq \pi^2 R + o(1)R.$$

$\square$

The following lemma of the area growth estimate of a minimal graph is well-known argument, and one can see [2] for the details

**Lemma 2.3.** *Let  $G = (\Omega, u)$  be a minimal graph in  $\mathbb{R}^3$ , the area of  $G \cap B^3(R)$  satisfies*

$$A(G \cap B^3(R)) \leq 3\pi R^2.$$

We are now ready to prove the main theorem.

*Proof of the theorem 1.1.* Let  $B^3(R)$  be the ball of radius  $R$  in  $\mathbb{R}^3$ . Since

$$\int_{G_i \cap B^3(R)} |\tilde{\nabla} u_i|^2 \leq \int_{G_i \cap \partial B^3(R)} u_i (\tilde{\nabla} u_i \cdot \frac{\partial}{\partial r})$$

where  $\tilde{\nabla}$  means the gradient operator on  $G_i$ , one has

$$\begin{aligned} 2\lambda_1^{\frac{1}{2}}(G_i \cap \partial B^3(R)) \int_{G_i \cap B^3(R)} |\tilde{\nabla} u_i|^2 &\leq 2\lambda_1^{\frac{1}{2}} \int_{G_i \cap \partial B^3(R)} u_i \cdot \frac{\partial u_i}{\partial r} \\ &\leq \lambda_1 \int_{G_i \cap \partial B^3(R)} u_i^2 + \int_{G_i \cap \partial B^3(R)} \left(\frac{\partial u_i}{\partial r}\right)^2 \\ &\leq \int_{G_i \cap \partial B^3(R)} |\tilde{\nabla} u_i|^2 + \int_{G_i \cap \partial B^3(R)} \left(\frac{\partial u_i}{\partial r}\right)^2 \\ &= \int_{G_i \cap \partial B^3(R)} |\tilde{\nabla} u_i|^2. \end{aligned}$$

Here  $\lambda_1^{\frac{1}{2}}(G_i \cap \partial B^3(R))$  denotes the first Dirichlet eigenvalue on  $G_i \cap \partial B^3(R)$ . As we know, in  $\mathbb{R}^3$  the following inequality holds:

$$\lambda_1^{\frac{1}{2}}(G_i \cap \partial B^3(R)) \geq \frac{\pi^2}{\ell^2(G_i \cap \partial B^3(R))}.$$

Therefore

$$\frac{\int_{G_i \cap \partial B^3(R)} |\tilde{\nabla} u_i|^2}{\int_{G_i \cap B^3(R)} |\tilde{\nabla} u_i|^2} \geq 2\lambda_1^{\frac{1}{2}} \geq \frac{2\pi}{\ell(\Gamma_i)},$$

where  $\Gamma_i := G_i \cap \partial B^3(R)$ . Thus we obtain

$$\sum_{i=1}^k \frac{\int_{G_i \cap \partial B^3(R)} |\tilde{\nabla} u_i|^2}{\int_{G_i \cap B^3(R)} |\tilde{\nabla} u_i|^2} \geq \sum_{i=1}^k \frac{2\pi}{\ell(\Gamma_i)}.$$

Notice that

$$k^2 \leq \left(\sum_{i=1}^k \ell(\Gamma_i)\right) \left(\sum_{i=1}^k \frac{1}{\ell(\Gamma_i)}\right).$$

According to the theorem 2.1, one has

$$\sum_{i=1}^k \ell(\Gamma_i) \leq \pi^2 R + o(1)R$$

for a sufficiently large radius  $R$ . Then it can be concluded

$$\sum_{i=1}^k \frac{\int_{G_i \cap \partial B^3(R)} |\tilde{\nabla} u_i|^2}{\int_{G_i \cap B^3(R)} |\tilde{\nabla} u_i|^2} \geq \frac{2\pi k^2}{R(\pi^2 + o(1))}. \quad (6)$$

Observing that

$$\int_{G_i \cap \partial B^3(r)} |\tilde{\nabla} u_i|^2 = \frac{\partial}{\partial r} \int_{G_i \cap B^3(r)} |\tilde{\nabla} u_i|^2. \quad (7)$$

From (6) and (7) to obtain

$$\ln\left(\prod_{i=1}^k \frac{\int_{G_i \cap B^3(R)} |\tilde{\nabla} u_i|^2}{\int_{G_i \cap B^3(R_0)} |\tilde{\nabla} u_i|^2}\right) \geq \frac{2\pi k^2}{\pi^2 + o(1)} \ln\left(\frac{R}{R_0}\right). \quad (8)$$

On the other hand, let  $(x, y, u_i(x, y))$  be a parametrization of  $G_i$ , then the induced metric on  $G_i$  is

$$ds^2 = (1 + (u_i)_x^2)dx^2 + 2(u_i)_x(u_i)_y dx dy + (1 + (u_i)_y^2)dy^2.$$

Hence

$$|\tilde{\nabla} u_i| = \sqrt{u_{x^i} u_{x^j} g^{ij}} = \sqrt{\frac{|\nabla u_i|^2}{1 + |\nabla u_i|^2}} \leq 1.$$

From it one can deduce

$$\prod_{i=1}^k \int_{G_i \cap B^3(R)} |\tilde{\nabla} u_i|^2 \leq A^k (G_i \cap B^3(R)) \leq (3\pi R^2)^k. \quad (9)$$

Combining (8) and (9) implies

$$\frac{2\pi k^2}{\pi^2 + o(1)} (\ln R - \ln R_0) \leq 2k \ln R + c_1.$$

Let  $R \rightarrow +\infty$  to have

$$k \leq \pi.$$

This inequality indicates that  $k \leq 3$ .

If  $k = 3$ , repeating the above process and using the refined length estimate in the theorem 2.1 provides

$$k \leq 2\sqrt{2}$$

which is a contradiction.

Thus  $k$  has to be at most 2.  $\square$

**Remark.** In [7], Tkachev has already proved the number of two dimensional disjointly supported minimal graphs is at most 3. Here a different approach can lead to a better estimate if assuming the Gauss curvature vanishes at infinity.

### 3 Proof of corollary 1.2

Let  $\pi_i : G_i \rightarrow \mathbb{R}^2$  be the natural projective map and  $B^2(R)$  be the ball of radius  $R$  in  $\mathbb{R}^2$ . By employing the same method in the proof of theorem 1.1 except for using a different region of integration  $\pi_i^{-1}(\Omega_i \cap B^2(R))$ , one can conclude

**Theorem 3.1.** Suppose  $\{(\Omega_i, u_i)\}_{i=1}^k$  is a set of disjointly supported minimal graphs in  $R^3$  where each  $\Omega_i$  is an open subset in  $R^2$ . If the gradient of each  $u_i$  is bounded by  $c$ , i.e.  $|\nabla u_i| \leq c$ , then the number  $k \leq 2\sqrt{1+c^2}$ .

*Proof.* : By a similar argument, one can obtain that

$$\sum_{i=1}^k \frac{\int_{\pi_i^{-1}(\Omega_i \cap \partial B^2(R))} |\tilde{\nabla} u_i|^2}{\int_{\pi_i^{-1}(\Omega_i \cap B^2(R))} |\tilde{\nabla} u_i|^2} \geq \frac{2\pi k^2}{\sum_{i=1}^k \ell(\Gamma_i)}.$$

where  $\Gamma_i := \pi_i^{-1}(\Omega_i \cap \partial B^2(R))$ . If one chooses the parameter  $(R \cos(\theta), R \sin(\theta), u_i(R \cos(\theta), R \sin(\theta)))$  of the curve  $\Gamma_i$  and assume  $|\nabla u_i| \leq c$ , then

$$\ell(\Gamma_i) = \int_{\theta_0}^{\theta_1} \sqrt{R^2 + [-(u_i)_x R \sin(\theta) + (u_i)_y R \cos(\theta)]^2} d\theta$$

$$\begin{aligned}
&\leq \int_{\theta_0}^{\theta_1} \sqrt{R^2 + [(u_i)_x^2 + (u_i)_y^2](R^2 \sin(\theta)^2 + R^2 \cos(\theta)^2)} d\theta \\
&\leq (\theta_1 - \theta_0) R \sqrt{1 + c^2}.
\end{aligned}$$

Since the minimal graphs are disjoint, so

$$\sum_{i=1}^k \ell(\Gamma_i) \leq 2\pi R \sqrt{1 + c^2}.$$

Then it can be concluded

$$\sum_{i=1}^k \frac{\int_{\pi_i^{-1}(\Omega_i \cap \partial B^2(R))} |\tilde{\nabla} u_i|^2}{\int_{\pi_i^{-1}(\Omega_i \cap B^2(R))} |\tilde{\nabla} u_i|^2} \geq \frac{k^2}{R \sqrt{1 + c^2}}. \quad (10)$$

Integrating (10), one will obtain

$$\ln \left( \prod_{i=1}^k \frac{\int_{\pi_i^{-1}(\Omega_i \cap \partial B^2(R))} |\tilde{\nabla} u_i|^2}{\int_{\pi_i^{-1}(\Omega_i \cap B^2(R_0))} |\tilde{\nabla} u_i|^2} \right) \geq \frac{k^2}{\sqrt{1 + c^2}} \ln \left( \frac{R}{R_0} \right). \quad (11)$$

On the other hand,

$$\begin{aligned}
\prod_{i=1}^k \int_{\pi_i^{-1}(\Omega_i \cap B^2(R))} |\tilde{\nabla} u_i|^2 &\leq A^k(\pi_i^{-1}(\Omega_i \cap B^2(R))) = \left( \int_{\Omega_i \cap B^2(R)} \sqrt{1 + |\nabla u|^2} \right)^k \\
&\leq (\sqrt{1 + c^2} \pi R^2)^k.
\end{aligned} \quad (12)$$

Combining (11) and (12), we have

$$\frac{k^2}{\sqrt{1 + c^2}} (\ln R - \ln R_0) \leq 2k \ln R + c_1.$$

Let  $R \rightarrow +\infty$  to have

$$k \leq 2\sqrt{1 + c^2}.$$

□

Obviously, corollary 1.2 follows from above theorem when each graph satisfies

$$|\nabla u_i| \rightarrow 0 \quad (|x| \rightarrow +\infty).$$

**Remark.** *J. Spruck has already proved the corollary 1.2 under the assumption of a certain decay rate of Gauss curvature at infinity [6]. However, here we do not need any kind of restrictions on Gauss curvature.*

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